2 Complex Integration

2.1 Integration along paths

Definition 2.1. Let $I = [a, b] \subset \mathbb{R}$ be a compact interval. A continuous map $\gamma : I \to \mathbb{C}$ is called a *curve* in $\mathbb{C}$. We denote the image of the curve γ by $|\gamma|$. If $\gamma(a) = \gamma(b)$, the curve is called *closed.* A curve $\gamma : I \to \mathbb{C}$ is called a *path* iff it is *piecewise continuously differentiable.* That is, there exist points $a = x_0 < x_1 < \cdots < x_n = b$ such that γ restricted to $[x_{k-1}, x_k]$ is continuously differentiable for all $k \in \{1, \ldots, n\}$.

Recall that *continuous differentiability* in a closed interval $[a, b]$ means differentiability in (a, b) such that the differential is continuous and has a continuous extension to $[a, b]$.

For the theory of integration along paths what is important in a path is its image in and in which direction this is retraced. In contrast, the concrete parametrization of a path via an interval $I \subset \mathbb{R}$ is not important. To make this more precise we define the concept of reparametrization of a path.

Definition 2.2. Let $\gamma : [a, b] \to \mathbb{C}$ and $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ be paths. We say that $\tilde{\gamma}$ is a *reparametrization* of γ iff there exists a monotonous, continuous and piecewise continuously differentiable map $\phi : [\tilde{a}, \tilde{b}] \to [a, b]$ with $\phi(\tilde{a}) = a$ and $\phi(\tilde{b}) = b$ and such that $\tilde{\gamma} = \gamma \circ \phi$.

We will be interested only in properties and usages of paths that are invariant under reparametrization. The first such property we consider is the length of a path. Intuitively it is quite clear what we mean by this. If a path $\gamma : [a, b] \to \mathbb{C}$ is a straight line

$$\gamma(t) := \frac{(b - t)x_1 + (t - a)x_2}{b - a}$$

with end points x_1 and x_2, then its length should be $|x_2 - x_1|$ where we use the standard Euclidean inner product on $\mathbb{C}$. In general, we can approximate a path by subdividing the interval on which it is defined and replacing the pieces of paths in subdivisions by straight lines. The length of the path should then be the limit of the sum of the lengths of these straight lines when we make the subdivisions arbitrarily fine. That this limit exists is due to the piecewise continuous differentiability property we have imposed. (The limit does not necessarily exist for arbitrary curves, even if their image is bounded.) The result is the following, which we state as a definition.

Definition 2.3. Let $\gamma : [a, b] \to \mathbb{C}$ be a path. The *length* of γ, denoted $l(\gamma)$ is defined by,

$$l(\gamma) := \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

Exercise 9. (a) Show that the definition indeed agrees with the result of the procedure described above. (b) Give an example of a curve that has bounded image, but no well defined length.

Exercise 10. Show that the length of a path is invariant under reparametrization. That is, show that if γ is a path and $\tilde{\gamma}$ is a reparametrization of γ, then $l(\gamma) = l(\tilde{\gamma})$.

Definition 2.4. Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be a continuous map. Let $\gamma : I \to \mathbb{C}$ be a path such that $|\gamma| \subset U$. We define the *complex integral* of f *along the path* γ as follows,

$$\int_\gamma f(z) \, \mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t. \tag{1}$$

To make sense of this definition we note that $t \mapsto f(\gamma(t))\gamma'(t)$ is a piecewise continuous function $I \to \mathbb{C}$ and is therefore bounded and integrable.

Proposition 2.5. *The complex integral is invariant under reparametrizations: Given an open set $U \subseteq \mathbb{C}$, a continuous function $f : U \to \mathbb{C}$, a path γ with $|\gamma| \subset U$ and a reparametrization $\tilde{\gamma}$ of γ. Then,*

$$\int_{\tilde{\gamma}} f(z) \, \mathrm{d}z = \int_\gamma f(z) \, \mathrm{d}z.$$

Proof. **Exercise.** □

Similarly to what we have seen in the context of the concept of derivative, the concept of integration introduced is quite similar to what we are familiar with in the case of $\mathbb{R}$ or $\mathbb{R}^n$. Nevertheless, again, there is an important difference that makes crucial use of the fact that the complex numbers form a field. If we were to discuss integration along paths in $\mathbb{R}^2$ weighted by path length, the formula to use would be almost identical to (1), with one important difference: γ' would be a 2×1-matrix and we would insert $|\gamma'(t)|$ instead of $\gamma'(t)$ on the right hand side. Decomposing $\gamma' = re^{\mathrm{i}\theta}$ the difference is that in the real case we would only put the absolute value r. We might think of the complex case as letting the direction of the curve (encoded in

θ) enter the integrand. As we shall see, this leads to a remarkable interplay between complex integral and derivative.

Suppose $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [b, c] \to \mathbb{C}$ are paths such that $\gamma_1(b) = \gamma_2(b)$. Then, we can form the *composite path* $\gamma_1 \cdot \gamma_2 : [a, c] \to \mathbb{C}$ in the obvious way. We have then,

$$\int_{\gamma_1 \cdot \gamma_2} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

Because of Proposition 2.5 we are usually interested in paths only up to reparametrization. That is, we consider two paths as equivalent if one is a reparametrization of the other. We may then talk about the composition of paths whenever the endpoint of the first coincides with the initial point of the second.

Given a path $\gamma : [0, 1] \to \mathbb{C}$ we may form the *opposite path* $\gamma^{-1} : [0, 1] \to \mathbb{C}$ given by $\gamma^{-1}(t) = \gamma(1 - t)$. Then clearly, $(\gamma^{-1})^{-1} = \gamma$. As is easy to see,

$$\int_{\gamma^{-1}} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

We also find that the integral of any function along $\gamma \cdot \gamma^{-1}$ vanishes. $\gamma \cdot \gamma^{-1}$ is called a *retracing*. Because the integral along a retracing vanishes, we consider a retracing as equivalent to the trivial path.

Exercise 11. The concept of reparametrization can be generalized to include some form of retracing. To this end remove the monotonicity condition from Definition 2.2. (a) Is the length of a path invariant under generalized reparametrization? (b) Is the complex integral along a path invariant under generalized reparametrization?

Proposition 2.6 (Transformation rule)**.** *Let $D \subseteq \mathbb{C}$ be a region, $g \in \mathcal{O}(D)$ such that $g' : D \to \mathbb{C}$ is continuous and γ a path with $|\gamma| \subset D$. Then, $g \circ \gamma$ is a path and for any continuous function $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is open and $|g \circ \gamma| \subset U$ we have,*

$$\int_{g \circ \gamma} f(z)\,\mathrm{d}z = \int_{\gamma} f(g(z))g'(z)\,\mathrm{d}z.$$

Proof. **Exercise.** □

Proposition 2.7. *Let $U \subseteq \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous, γ be a path with $|\gamma| \subset U$. Set $\|f\|_\gamma := \sup_{z \in |\gamma|} |f(z)|$. Then,*

$$\left| \int_{\gamma} f(z)\,\mathrm{d}z \right| \leq |f|_\gamma l(\gamma).$$

Proof. **Exercise.** □

Proposition 2.8. *Let $U \subseteq \mathbb{C}$ be open and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of continuous functions $f_n : U \to \mathbb{C}$ converging uniformly. Let γ be a path in U. Then,*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,\mathrm{d}z = \int_\gamma \lim_{n\to\infty} f_n(z)\,\mathrm{d}z.$$

Proof. **Exercise.**[Hint: Use Proposition 2.7.] □

2.2 Integrable Functions

Definition 2.9. Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ be continuous. If $F \in \mathcal{O}(D)$ such that $F' = f$, then F is called a *primitive* of f. f is called *integrable* in D if there exists such a primitive.

Theorem 2.10. *Let $D \subseteq \mathbb{C}$ be a region, $f : D \to \mathbb{C}$ be continuous and $F : D \to \mathbb{C}$. Then, F is a primitive of f iff for every path $\gamma : [a,b] \to D$*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Suppose F is a primitive of f. Assume without loss of generality that γ is continuously differentiable everywhere. Then, using the chain rule,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b F'(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b (F\circ\gamma)'(t)\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a)).$$

Conversely, suppose that F satisfies the stated formula for every path γ in D. Let $z \in D$ and choose $r > 0$ such that $B_r(z) \subseteq D$. For $\xi \in B_r(0)$ let $\gamma_\xi : [0,1] \to \mathbb{C}$ be the path $\gamma_\xi(t) := z + t\xi$. By assumption,

$$F(z+\xi) - F(z) = \int_{\gamma_\xi} f(\zeta)\,\mathrm{d}\zeta = \int_0^1 f(z+t\xi)\xi\,\mathrm{d}t$$

For $\xi \neq 0$ we get,

$$\frac{F(z+\xi) - F(z)}{\xi} = \int_0^1 f(z+t\xi)\,\mathrm{d}t.$$

The right hand side of this expression converges to $f(z)$ when $|\xi| \to 0$ since,

$$\begin{aligned}\left|\left(\int_0^1 f(z+t\xi)\,\mathrm{d}t\right) - f(z)\right| &\le \int_0^1 |f(z+t\xi) - f(z)|\,\mathrm{d}t \\ &\le \sup_{\zeta\in B_{|\xi|}(0)} |f(z+\zeta) - f(z)|,\end{aligned}$$

where the right hand side expression converges to zero for $|\xi| \to 0$ by continuity of f. Thus, F is complex differentiable at z with the differential being $F'(z) = f(z)$. This completes the proof. $\square$

Proposition 2.11. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ be continuous. Then, f is integrable in D iff for every closed path γ in D we have:*

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof. If f is integrable, then by Theorem 2.10 the integral along any close path must be zero. Conversely, suppose the integral of f along any closed path is zero. Choose $z_0 \in D$ arbitrarily. Define

$$F(z) := \int_{\gamma_z} f(z)\,\mathrm{d}z,$$

where $\gamma_z : [a, b] \to D$ is a path such that $\gamma_z(a) = z_0$ and $\gamma_z(b) = z$. Such a path always exists by the path-connectedness of D. Also, the definition of $F(z)$ is well, since any other path with the same end points must yield the same value by assumption. $F : D \to \mathbb{C}$ defined in this way satisfies the assumption of Theorem 2.10 and is thus a primitive of f. $\square$

Definition 2.12. Let $D \subset \mathbb{C}$ be a region. We call D *star-shaped* with *center* $z_0 \in D$ iff for every element $z \in D$ the path $\gamma : [0, 1] \to \mathbb{C}$ given by $\gamma(t) := z_0 + t(z - z_0)$ lies entirely in D.

A triangle Δ is a closed subset of $\mathbb{C}$ with the shape of a triangle. Its boundary $\partial\Delta$ is the union of three straight line segments. We also denote by $\partial\Delta$ a closed path that traces the boundary of the triangle once in positive (i.e., counter-clockwise) direction.

Proposition 2.13. *Let $D \subseteq \mathbb{C}$ be a star-shaped region with center z_0. Let $f : D \to \mathbb{C}$ be continuous. Then, f is integrable in D iff for every triangle Δ in D with z_0 a corner,*

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0.$$

Proof. If f is integrable, we obtain the required implication as a special case of Proposition 2.11. Conversely, we show that f is integrable if the integral along all triangles in D with one vertex in z_0 vanishes. We define a function $F : D \to \mathbb{C}$ as follows. Let $z \in D$ and define the path $\gamma_z : [0, 1] \to \mathbb{C}$ by

$\gamma(t) := z_0 + t(z - z_0)$. Since D is star-shaped with center z_0, the path γ_z lies entirely in D. Then set,

$$F(z) := \int_{\gamma_z} f(z)\,\mathrm{d}z.$$

Fix $z \in D$. By star-shapedness of D there exist $r > 0$ such that $B_r(z) \subseteq D$ and for all $\zeta \in B_r(z)$ the path γ_ζ lies entirely in D. For all $\xi \in B_r(0)$ set $\tilde{\gamma}_\xi : [0,1] \to \mathbb{C}$ to be the path $\tilde{\gamma}_\xi(t) = z + t\xi$. Then, by assumption,

$$F(z+\xi) - F(z) = \int_{\gamma_{z+\xi}} f(z)\,\mathrm{d}z - \int_{\gamma_z} f(z)\,\mathrm{d}z = \int_{\tilde{\gamma}_\xi} f(z)\,\mathrm{d}z,$$

and we may proceed as in the proof of Theorem 2.10 to show that F is a primitive of f at z. This completes the proof. $\square$

Proposition 2.14 (Integral Lemma of Goursat). *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ and $\Delta \subset D$ a triangle. Then,*

$$\int_{\partial\Delta} f(z)\,\mathrm{d}z = 0.$$

Proof. We produce a sequence of triangles $\{\Delta_n\}_{n\in\mathbb{N}}$ with $\Delta_n \subset D$ by iteration. Set $\Delta_1 := \Delta$. To produce Δ_{n+1} from Δ_n proceed as follows. Subdivide Δ_n into four triangles $\Delta_{n,1}, \ldots, \Delta_{n,4}$ by subdividing each of its sides into two pieces of equal length. Now choose $k \in \{1,2,3,4\}$ such that the absolute value

$$\left| \int_{\Delta_{n,k}} f(z)\,\mathrm{d}z \right|$$

is maximized and set $\Delta_{n+1} := \Delta_{n,k}$. This defines a sequence of triangles. Note that the intersection $\bigcap_{n\in\mathbb{N}} \Delta_n$ is a single point $z_0 \in D$.

By the addition property of the integral along paths we have for every $n \in \mathbb{N}$ the identity

$$\int_{\partial\Delta_n} f = \int_{\partial\Delta_{n,1}} f + \int_{\partial\Delta_{n,2}} f + \int_{\partial\Delta_{n,3}} f + \int_{\partial\Delta_{n,4}} f.$$

By the maximality condition of our construction, this implies, for all $n \in \mathbb{N}$,

$$\left| \int_{\partial\Delta_n} f \right| \leq 4 \left| \int_{\partial\Delta_{n+1}} f \right|, \tag{2}$$

and thus,

$$\left|\int_{\partial\Delta} f\right| \le 4^{n-1}\left|\int_{\partial\Delta_n} f\right|.$$

For the circumference of the triangles we obtain the relation,

$$l(\partial\Delta_n) = \frac{1}{2^{n-1}} l(\partial\Delta). \tag{3}$$

Now set $\epsilon > 0$ arbitrarily and choose $r > 0$ such that $B_r(z_0) \subseteq D$ and

$$|g(z)| < \epsilon|z - z_0|, \quad \text{where} \quad g(z) := f(z) - f(z_0) - f'(z_0)(z - z_0)$$

for all $z \in B_r(z_0)$. (This is possible since f is complex differentiable at z_0.) Now fix $n \in \mathbb{N}$ such that $\Delta_n \subset B_r(z_0)$. Note that the constant function and the identity function are integrable so that with Proposition 2.11 we have,

$$\int_{\partial\Delta_n} f(z)\,\mathrm{d}z = \int_{\partial\Delta_n} \big(f(z_0) + f'(z_0)(z - z_0) + g(z)\big)\,\mathrm{d}z = \int_{\partial\Delta_n} g(z)\,\mathrm{d}z.$$

Using the estimate of Proposition 2.7, and (3),

$$\left|\int_{\partial\Delta_n} f\right| \le \|g\|_{\partial\Delta_n} l(\partial\Delta_n) < \frac{\epsilon}{2}\, l(\partial\Delta_n)^2 = \frac{\epsilon}{2^{2n-1}} l(\partial\Delta)^2.$$

On the other hand, combining this with (2) we get,

$$\left|\int_{\partial\Delta} f\right| < \frac{\epsilon}{2} l(\partial\Delta)^2.$$

Since ϵ was arbitrary, we conclude that the integral of f along $\partial\Delta$ vanishes. □

Corollary 2.15. *Let $D \subseteq \mathbb{C}$ be a star-shaped region and $f \in \mathcal{O}(D)$. Then, f is integrable in D.*

Proof. This is obtained by combining Proposition 2.13 with Proposition 2.14. □

We arrive at the important conclusion that a holomorphic function is integrable (in star-shaped regions). Soon we will see that the converse is also true: An integrable function is holomorphic.

Exercise 12. Let $D := \mathbb{C} \setminus [0, 1]$. Show that $f(z) := \frac{1}{z(z-1)}$ is integrable in D. [Hint: Observe that $f(z) = \frac{1}{z-1} - \frac{1}{z}$ and use primitives for the summands. Be careful about the domain of definition.]

Exercise 13. Let $D \subseteq \mathbb{C}$ be a region and $\{f_n\}_{n\in\mathbb{N}}$ a sequence of continuous integrable functions converging uniformly to a function $f : D \to \mathbb{C}$. Show that f is integrable in D.

Exercise 14. Let $D_1, D_2 \subseteq \mathbb{C}$ be regions such that $D_1 \cap D_2$ is connected. Let $f : D_1 \cup D_2 \to \mathbb{C}$ be continuous. (a) Show that if f is integrable in D_1 and also integrable in D_2, then f is integrable in $D_1 \cup D_2$. (b) Give a counter example in the case when the connectedness condition is removed.